

On the Calculation of Scalar Mesons in Functional Nonlinear Spinor Theory

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The formulation of nonlinear spinor theory in functional space is used for the calculation of scalar meson masses. The second order equation used, requires an explicit angular momentum reduction. For illustration, this method is also applied to the first order equation. In second order, we get an integral equation of the Bethe-Salpeter type which is solved in Fredholm approximation.

1. Introduction

Nonlinear spinor theory is an attempt to formulate a unified framework of high energy phenomena. In the original version of Heisenberg and co-workers¹, the basic equation was an operator equation for the field operators. Because of the assumed indefinite metric of the state space, the field operators act in a space of unknown structure. So they are almost useless for carrying out dynamical calculations, but merely serve for stating the symmetries of the theory.

In order to remedy this defect of the theory, Stumpf² proposed a functional quantum theory. The field equation is replaced by an equation for the generating functional of the transition matrix elements. These are well-defined mathematical objects even in the presence of an indefinite metric. All physical information has to be extracted from these functionals. The aim of the theory is the calculation of cross-sections for the scattering of clusters, which is in progress^{3,4}. But the first step is the construction of the eigenfunctionals of the clusters and the calculation of the corresponding eigenvalues. In this paper, we therefore continue the calculation of mass eigenvalues of mesons which was started in⁵, cited as in I in the following. There we considered vector mesons whereas we now turn to the case of scalar particles. Of course eigenvalues of meson masses have been obtained already much earlier, see for example^{1,6}. But in these papers only the lowest order approximation was considered which yields an almost trivial equation. Because of the contact type of interaction involved, it does especially not require the angular momentum decomposition which makes things so tedious in higher orders. The second order approximation was considered in⁷ without performing the angular momen-

tum decomposition. But the claim to get only scalar mesons by the assumed symmetry properties of the wave-function is easily seen to be not correct.

In second order, the resulting eigenvalue equation is a true integral equation in contrast to the first order one. For its solution we use the first Fredholm approximation as already in I. Considerations on the scalar Bethe-Salpeter equation⁸ show that the correctness of the approximation depends heavily on the mass of the exchanged particles compared to the external ones. In nonlinear spinor theory, a continuum of particles is exchanged, and it is difficult to judge on the validity of the approximation. Therefore it is desirable to have a different approximation scheme. Work in this direction is in progress, and we hope to report on it soon.

The present paper is organized as follows. In Sect. 2 we collect the basic equations, derived at length in I. In Sect. 3 the symmetries of the equation are discussed, especially the discrete symmetry of CP-conjugation. The lowest order equation is discussed in Sect. 4 in order to explain the method of solution on a comparatively simple example. The second order is considered in Section 5. In Sect. 6 we give some final remarks. Technical details are discussed in the appendices.

2. Fundamentals

We give here a short summary of the basic equations of functional nonlinear spinor theory. For a more detailed account see I. The equation for the generating functional $|\mathcal{Z}_a(j)\rangle$ reads for noncanonical quantization:

$$\{D_{a\beta}\partial_\beta - V_{a\beta\gamma\delta}(\partial_\beta\partial_\gamma\partial_\delta - 3F_{\beta\gamma}\partial_\delta)\}|\mathcal{Z}_a(j)\rangle = 0. \quad (2.1)$$



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Every index is meant as a triple of discrete indices $\alpha_1 \alpha_2 \alpha_3$ and a continuous variable x . A summation and integration over repeated indices is understood. $D_{\alpha\beta}$ is a first order differential operator and $V_{\alpha\beta\gamma\delta}$ a local vertex. Their explicit form depends on the representation used. We choose the Hermitian Weyl representation, since the Hermitian representation has several advantages. It might be desirable to use the Dirac form because of parity, but if the chiral invariance of the equation is not broken by a mass term, we will get identical results in the Dirac form, since the states will occur as degenerate parity doublets⁹. The explicit form of $D_{\alpha\beta}$ and $V_{\alpha\beta\gamma\delta}$ is given in I.

Next we invert $D_{\alpha\beta}$ with the causal Green's function:

$$D_{\alpha\beta} G_{\beta\gamma} = \delta_{\alpha\gamma} \quad (2.2)$$

and transform to normalordered functionals by

$$|\mathfrak{T}_a(j)\rangle = \exp\left\{-\frac{1}{2} j_a F_{a\beta} j_\beta\right\} |\Phi_a(j)\rangle. \quad (2.3)$$

$F_{a\beta}$ is the two-point function of the theory which is assumed to be regularized because of the indefinite metric of the state space. Introducing

$$d_a = \partial_a - F_{a\beta} j_\beta \quad (2.4)$$

the equation for $|\Phi_a(j)\rangle$ is

$$\partial_a |\Phi_a(j)\rangle = O_a |\Phi_a(j)\rangle \quad (2.5)$$

with the operator

$$O_a = F_{a\beta} j_\beta + G_{a\beta} V_{\beta\gamma\delta\epsilon} (d_\gamma d_\delta d_\epsilon - 3 F_{\gamma\delta} d_\epsilon). \quad (2.6)$$

Evidently, in (2.5) one coordinate is singled out. In order to get fully antisymmetrical equations one has to remove this by symmetrizing (2.5). This can be done in different ways. First we can use an indefinite functional integral as proposed in¹⁰:

$$\int \delta j_a |\mathfrak{T}_a(j)\rangle = |\mathfrak{S}(j)\rangle : \Leftrightarrow \partial_a |\mathfrak{S}(j)\rangle = |\mathfrak{T}_a(j)\rangle. \quad (2.7)$$

We will use it in Section 4. Next we may apply the generator of a symmetry group to (2.5), for example the momentum operator:

$$\mathfrak{P}_\mu |\Phi_a(j)\rangle = \mathfrak{D}_\mu |\Phi_a(j)\rangle = P_\mu |\Phi_a(j)\rangle. \quad (2.8)$$

Because the state has a definite momentum P_μ , we get the second equality in (2.8). Iterating (2.8), we get the equation proposed in³ and used already in I. We will use it in Sect. 5 for the calculation of scalar mesons. In both cases, we have now an

equation of the type

$$|\Phi_a(j)\rangle = \mathfrak{G} |\Phi_a(j)\rangle \quad (2.9)$$

with

$$\mathfrak{G} = \begin{cases} \int \delta j_a O_a & \text{for (2.7),} \\ \mathfrak{D}^\mu \mathfrak{D}_\mu / P^2 & \text{for (2.8).} \end{cases} \quad (2.10)$$

Note that in the second case the first term in (2.6) does not contribute to \mathfrak{D}_μ .

Because of the quantum numbers of the state, the functional will start with a lowest order function $|\Phi_\varrho(j)\rangle$. Defining the projection operators \mathbf{P}_ϱ by

$$\mathbf{P}_\varrho |\Phi_a(j)\rangle = |\Phi_\varrho(j)\rangle \quad (2.11)$$

and

$$\mathbf{\Pi}_\varrho = \sum_{r=\varrho+1}^{\infty} \mathbf{P}_r \quad (2.12)$$

we get from (2.10) an equation for $|\Phi_\varrho(j)\rangle$:

$$|\Phi_\varrho(j)\rangle = \mathbf{P}_\varrho \mathfrak{G} [1 - \mathbf{\Pi}_\varrho \mathfrak{G}]^{-1} |\Phi_\varrho(j)\rangle. \quad (2.13)$$

We expand the inverse in a Neumann series and retain only the first order term, yielding:

$$|\Phi_\varrho(j)\rangle = \mathbf{P}_\varrho \mathfrak{G} |\Phi_\varrho(j)\rangle. \quad (2.14)$$

This corresponds to a perturbation expansion of the two-particle irreducible kernel of the Bethe-Salpeter equation and keeping only the ladder approximation. It would be highly desirable to do something better at this point.

For mesons we have $\varrho=2$. Equation (2.14) may be represented in a graphical notation which is explained in I. For the "integral average" (2.7) we have:

$$\mathfrak{D} = \frac{1}{2} (\text{X} - \text{Y}) \quad (2.15)$$

and for the "mass average" (2.9)

$$\begin{aligned} \mathbf{P}^* \mathfrak{D} = 18 & \left(\text{A} - \text{B} \right) \\ & + 36 \left(\text{C} - \text{D} \right) \\ & + 36 \left(\text{E} \right) + 18 \left(\text{F} + \text{G} \right) \end{aligned} \quad (2.16)$$

Because of the iteration of (2.8) we have in this case already from the first order term in the expansion of (2.13) a nontrivial equation. As in I we will consider only the first row of (2.16) though the inclusion of the self-energy graphs might be important. In effect, they will shift the zero mass pole in G to a finite value. An interesting investigation in this direction has been made in¹¹. (2.15) will be solved in Sect. 4, (2.16) in Section 5.

3. Angular Momentum and Discrete Symmetries

The solutions of the dynamical Eqs. (2.15), (2.16) may be classified according to the quantum numbers of the state. The internal symmetries are given by the baryon number gauge group and the isospin group. Their reduction has been described in detail in I.

Much more intricate is the reduction with respect to the Lorentz group because in this case also the spatial part of the wave functions is involved. It has been considered in I, too, but we want to review it here shortly in order to demonstrate the difference to the nonrelativistic case arising from the fourth coordinate.

From the symmetry conditions in functional space¹² or from the definition of the wave function as matrix elements of field operators¹³, we have in the rest system the eigenvalue conditions:

$$\begin{aligned} \mathbf{J}^2 \varphi(p) &:= \sum_{i=1}^3 [L_i(p) + \sigma_i \otimes 1 - 1 \otimes \sigma_i^T]^2 \varphi(p) \\ &= j(j+1) \varphi(p), \quad (3.1) \\ J_3 \varphi(p) &:= [L_3(p) + \sigma_3 \otimes 1 - 1 \otimes \sigma_3^T] \varphi(p) \\ &= j_3 \varphi(p). \end{aligned}$$

$L_i(p)$ are the components of the "orbital angular momentum" and σ_i the usual Pauli matrices. If we denote

$$S_i = \sigma_i \otimes 1 - 1 \otimes \sigma_i^T, \quad (3.2)$$

\mathbf{S} and \mathbf{L} form a commuting set of angular momentum operators. So the eigenstates of $\mathbf{J}^2 = (\mathbf{L} + \mathbf{S})^2$ can be constructed with the usual Clebsch-Gordan algebra. The eigenstates of \mathbf{S}^2 and S_3 are the Pauli matrices in the spherical basis, $\bar{\sigma}_{ss_3}$ (see I), and those of \mathbf{L}^2 and L_3 are the wellknown spherical harmonics $Y_{ll_3}(\Omega_p)$. So the eigenstates of \mathbf{J}^2 and J_3 are given by

$$\varphi_{jj_3}^{sl}(p) = \sum_{\substack{ss_3 \\ ll_3}} \langle s s_3 l l_3 | j j_3 \rangle \bar{\sigma}_{ss_3} Y_{ll_3}(\Omega_p) \varphi_{jj_3}^{sl}(p_0, p_1) \quad (3.3)$$

with $p_1 = |\mathbf{p}|$. The convention of Clebsch-Gordan coefficients is taken from¹⁴. The reduced matrix elements $\varphi_{jj_3}^{sl}(p_0, p_1)$ have to be calculated from the dynamical equation. The inverse of (3.3) is easily obtained with the help of well-known orthogonality relations:

$$\begin{aligned} \varphi_{jj_3}^{sl}(p_0, p_1) &= \sum_{s_3 l_3} \langle s s_3 l l_3 | j j_3 \rangle \int d\Omega_p Y_{ll_3}^*(\Omega_p) \\ &\quad \cdot \text{Tr}[\sigma_{ss_3} \varphi_{jj_3}(p)]. \quad (3.4) \end{aligned}$$

It has to be noted that the wave function depends only on j and j_3 . Contrary to the nonrelativistic case, \mathbf{L}^2 and \mathbf{S}^2 are no longer separately constants of the motion. Since the bound state is a one particle state, it is even not justified to speak of "orbital angular momentum". This would incorporate the idea of two particles circling around a common center of mass.

(3.3) as it stands is only valid in the rest frame $P = (m, 0, 0, 0)$. But if we will use φ_{jj_3} as a cluster wave function in scattering calculations, we have to know its form in an arbitrary frame. This is easily achieved by applying a Lorentz boost and amounts simply in some substitutions in the final formula, as is discussed in¹⁵.

Next we investigate the restriction due to the discrete symmetries. Since we are dealing with Weyl spinors we have only the CP-transformation. The eigenvalue equation corresponding to (3.1) reads in this case:

$$C \mathcal{P} \varphi(p) = c_o \varphi^T(-p_0, \mathbf{p}) c_o = \eta_{CP} \varphi(p) \quad (3.5)$$

and is again valid only in the rest frame (see also¹⁶). Strictly speaking, it applies only to the neutral members of the isospin multiplets. For the charged particles it has to be replaced by the GP-transformation.

If we substitute (3.3) into (3.5), we get the following symmetry properties of the reduced matrix elements:

$$\varphi_{jj_3}^{sl}(-p_0, p_1) = -(-1)^s \eta_{CP} \varphi_{jj_3}^{sl}(p_0, p_1). \quad (3.6)$$

So the number of components is not further reduced by the CP-invariance. It simply states that symmetric amplitudes $\varphi_{jj_3}^{0j}$ are coupled to antisymmetric $\varphi_{jj_3}^{1l}$ and vice versa.

This is in marked contrast to the scattering situation when the two external particles are on their mass-shell. In this case we have $p_0 = 0$, and one of the amplitudes will vanish. So one may speak of singlet and triplet scattering lengths etc. In the bound state situation, it will only be an approximation to neglect one amplitude, valid in the case of weak binding, e. g. for positronium. In the strong binding case it will make no sense at all. Whether it is a valid approximation in nonlinear spinor theory to neglect one amplitude (as it was done in I), has to be checked by explicit calculations.

4. First Order Approximation

We apply now the method of angular momentum reduction developed in the last section to the lowest order Equation (2.15). Of course we could handle this equation much more easily with the Rarita-Schwinger formalism because of the contact interaction. This was already done in¹ and¹⁰. We only want to exemplify the method of calculation in a comparatively simple case. After reduction of the baryon number gauge group and the isospin group, Eq. (2.15) reads explicitly:

$$\varphi(p) = [3 - T(T+1)] \frac{l^2}{(2\pi)^4} [G^\mu(p_-) F^\nu(p_+) + G^\nu(p_+) F^\mu(p_-)] \cdot \bar{\sigma}_\mu \sigma^\lambda \int d^4 q \varphi(q) \sigma_\lambda \bar{\sigma}_\nu. \quad (4.1)$$

T is the isospin of the meson with the values 0 and 1. With $p_\pm = p \pm \frac{1}{2}P$ the functions F and G are defined by

$$G^\mu(p) = p^\mu g(p) = p^\mu / (p^2 + i\varepsilon), \quad (4.2)$$

$$F^\mu(p) = p^\mu f(p) = i\kappa^4 p^\mu / [(p^2 + i\varepsilon)^2 (p^2 - \kappa^2 + i\varepsilon)].$$

l^2 is the coupling constant, calculated self-consistently in the nucleon calculation¹⁰.

Now we proceed to the angular momentum reduction: We substitute (3.3) for the wave function on both sides of the equation and project out the reduced matrix element with the help of (3.4). We can do the trace of the Pauli matrices, the angular integration, and the Clebsch-Gordan algebra without specifying the quantum numbers. The relevant formulas are collected in appendix I. The resulting equations for the reduced matrix elements are:

$$\varphi_{00}^{00}(p_0, p_1) = [3 - T(T+1)] \frac{l^2}{(2\pi)^4} h(p) [4\pi p_0^2 + 4\pi p_1^2 - \pi m^2] \int dq_0 dq_1 q_1^2 \varphi_{00}^{00}(q_0, q_1)$$

$$\varphi_{00}^{11}(p_0, p_1) = [3 - T(T+1)] \frac{l^2}{(2\pi)^4} h(p) [8\pi p_0 p_1] \int dq_0 dq_1 q_1^2 \varphi_{00}^{00}(q_0, q_1) \quad (4.3)$$

$$\varphi_{1j}^{01}(p_0, p_1) = [3 - T(T+1)] \frac{l^2}{(2\pi)^4} h(p) \left[-\frac{8\pi}{\sqrt{3}} p_0 p_1 \right] \int dq_0 dq_1 q_1^2 \varphi_{1j}^{10}(q_0, q_1)$$

$$\varphi_{1j}^{10}(p_0, p_1) = [3 - T(T+1)] \frac{l^2}{(2\pi)^4} h(p) [-4\pi p_0^2 + \frac{4}{3}\pi p_1^2 + \pi m^2] \int dq_0 dq_1 q_1^2 \varphi_{1j}^{10}(q_0, q_1)$$

$$\varphi_{1j}^{11}(p_0, p_1) = [3 - T(T+1)] \frac{l^2}{(2\pi)^4} h(p) \left[\frac{8\pi}{\sqrt{6}} m p_1 \right] \int dq_0 dq_1 q_1^2 \varphi_{1j}^{10}(q_0, q_1) \quad (4.4)$$

$$\varphi_{1j}^{12}(p_0, p_1) = [3 - T(T+1)] \frac{l^2}{(2\pi)^4} h(p) \left[8\pi \frac{\sqrt{2}}{3} p_1^2 \right] \int dq_0 dq_1 q_1^2 \varphi_{1j}^{10}(q_0, q_1)$$

and $\varphi_{jj}^{sl} \equiv 0$ for $j > 1$ because of the contact interaction. We have used the abbreviation

$$h(p) = g(p_-) f(p_+) + g(p_+) f(p_-). \quad (4.5)$$

For $j=0$ we have only two equations because of the Clebsch-Gordan coefficients, whereas for all higher angular momenta there are four equations (in principle). Only the first equation of (4.3) and the second one of (4.4) are true integral equations. All others are simply recursion formulas for the reduced matrix elements and have not to be considered in calculating the eigenvalues. Furthermore we get nontrivial solutions only for even functions φ_{jj}^{j0} ($j=0,1$) of p_0 . From the symmetry properties (3.6), this implies the following eigenvalues under CP-transformations:

$$\begin{aligned} j=0: & \eta_{CP} = -1, & \text{Particles } \pi, \eta; \\ j=1: & \eta_{CP} = +1, & \text{Particles } \rho, \omega. \end{aligned}$$

We now turn to the solution of the nontrivial integral equations. They are seen to be of a very simple type: The kernel is separable and does even not depend on one variable. In this case the Fredholm approximation is known to be the exact solution. Hence we get the eigenvalues by calculating the trace of the kernel. We will consider only the $j=0$ case in detail, the other one runs on completely the same pattern.

So we have to solve the following equation:

$$1 = [3 - T(T+1)] \frac{l^2}{(2\pi)^4} \int_{-\infty}^{\infty} dp_0 \int_0^{\infty} dp_1 p_1^2 h(p) [4\pi p_0^2 + 4\pi p_1^2 - \pi m^2] \quad (4.6)$$

or

$$1 + [3 - T(T+1)] \frac{1}{4} \left(\frac{\kappa l}{2\pi} \right)^2 q_0(\lambda) = 0 \quad (4.7)$$

with $\lambda = m^2/\kappa^2$ and

$$q_0(\lambda) = -\frac{1}{\pi \kappa^2} \int_{-\infty}^{\infty} dp_0 \int_0^{\infty} dp_1 p_1^2 h(p) [4p_0^2 + 4p_1^2 - m^2]. \quad (4.8)$$

The integral in (4.8) can be done in many ways. The simplest is to restore the original four-dimensional integration and then to use the well-known methods for calculating Feynman-integrals. In this way one sees that (4.8) yields the result known from e. g. ¹. But this method does not work for the nonlocal graph, to be considered in the next section, because of additional logarithmic factors from the angular integration. There we have to use the method described in I, and we will demonstrate it on (4.8).

First we do the p_0 -integration by the residual method. This yields:

$$q_0(\lambda) = -8 \int_0^{\infty} dp_1 p_1^2 \left[\frac{1}{m^2 - (\sqrt{p_1^2 + \kappa^2} + p_1)^2} \frac{1}{\sqrt{p_1^2 + \kappa^2}} + \frac{1}{p_1(4p_1^2 - m^2)} \right]. \quad (4.9)$$

In order to perform the p_1 -integration, we split the integral into two parts in an obvious way and substitute in the first part: $p_1 = \kappa x/(2\sqrt{1-x})$ and in the second one: $x = 4p_1^2/\kappa^2$. This removes the square roots from the first part and yields:

$$q_0(\lambda) = -\int_0^1 dx \frac{x^2}{1-x} \frac{1}{(1-x)\lambda - 1} - \int_0^{\infty} dx \frac{1}{x-\lambda}. \quad (4.10)$$

The two integrals are now separately divergent at the upper limit but the divergences cancel exactly. We could have avoided this by regularizing each term in (4.9) before performing the substitution.

The first term has a pole for $\lambda > 1$ and the second one for $\lambda > 0$. This yields cuts in the function $q_0(\lambda)$, starting at $\lambda = 1$ and $\lambda = 0$, respectively, if one remembers the $i\varepsilon$ -prescription in (4.2). These cuts are caused by the coincidence of the poles in F and G . Both of them are unphysical, the physical cut starting at $\lambda = 4$. To avoid this difficulty, one has to use the principal value prescription in evaluating (4.10), as discussed in detail in ¹.

Having this in mind, the integrations in (4.10) are done easily, yielding:

$$q_0(\lambda) = -\frac{1}{\lambda} + \ln|\lambda| - \frac{(1-\lambda)^2}{\lambda^2} \ln|1-\lambda|. \quad (4.11)$$

Fixing the coupling constant now to the value obtained from the nucleon calculation ¹:

$$\left(\frac{\kappa l}{2\pi} \right)^2 = 1.034 \quad (4.12)$$

we get from (4.7) the mass eigenvalues:

$$\begin{aligned} T=0(\eta): \quad m &= 0.92 \kappa \quad (\text{exp.: } m = 0.58 \kappa), \\ T=1(\pi): \quad m &= 0.31 \kappa \quad (\text{exp.: } m = 0.15 \kappa). \end{aligned}$$

Similarly we get from the second equation of (4.4) the eigenvalues of the vector mesons:

$$\begin{aligned} T=0(\omega): \quad m &= 0.33 \kappa \quad (\text{exp.: } m = 0.82 \kappa), \\ T=1(\rho): \quad m &= 0.10 \kappa \quad (\text{exp.: } m = 0.78 \kappa). \end{aligned}$$

Whether these solutions correspond to physical particles or to ghost states, has to be decided by calculating the norm of the states. In ¹⁷ the Bethe-Salpeter normalization was applied which showed the vector particles to be ghost states. But this is clearly inadequate because of the indefinite metric. One should use a normalization in functional space as it was done in ⁴. There it was found that all solutions correspond to physical particles, i. e. they have positive norm. By a similar, but slightly different method it was found in ¹⁸, too, that the scalar particles are physical. The vector particles were not considered there.

5. Second Order Approximation

Having explained the solution procedure on the first order equation, we now turn to the much more complicated Equation (2.16). Since a similar equation is obtained if we take into account the second term in the Neumann expansion of (2.13) for the integral average (2.7), we will call it the second order approximation. As it was discussed in Sect. 2, we will consider only the first row of the equation in order to get a first insight.

The first term is again a local graph as that discussed in Section 4. We have to calculate additionally only the Feynman integral for the inner loop. This was done already in I, we quote here only the result for $j=0$. It reads

$$1 + [3 - T(T+1)]^2 \frac{1}{16} \left(\frac{\kappa l}{2\pi} \right)^4 q_K(\lambda) = 0 \quad (5.1)$$

with

$$q_K(\lambda) = -\frac{4}{3} \left[\frac{1}{\lambda^2} - \frac{5}{2\lambda} + \ln|\lambda| + \frac{(1-\lambda)^3}{\lambda^3} \ln|1-\lambda| \right]^2. \quad (5.2)$$

It will contribute only for $\eta_{CP} = -1$ because only in this case one will get nontrivial wave functions.

The really new feature of (2.16) is provided by the second term which is a nonlocal graph. So it will give rise to a true integral equation, similar to the ladder approximation of the Bethe-Salpeter equation. After performing the reduction of baryon number and isospin, the equation reads explicitly:

$$\begin{aligned} P^2 \varphi(p) = & -\frac{16l^4}{(2\pi)^8} [3 - \tfrac{1}{2}T(T+1)] [G_\mu^o(p_+) F_\tau(p_-) + G_\tau^o(p_-) F_\mu(p_+)] \\ & \cdot \int d^4q M_{\lambda_{\rho\nu}}(p-q) \bar{\sigma}^\mu \sigma^\kappa \bar{\sigma}^\lambda \sigma^\eta \varphi(q) \sigma_\kappa \bar{\sigma}^\nu \sigma_\eta \bar{\sigma}^\tau \\ & + \frac{8l^4}{(2\pi)^8} [3 - T(T+1)] [G_\mu^o(p_+) F_\tau(p_-) + G_\tau^o(p_-) F_\mu(p_+)] \\ & \cdot \int d^4q M_{\lambda_\rho}^\lambda(p+q) \bar{\sigma}^\mu c_\sigma \varphi^T(q) c_\sigma \bar{\sigma}^\tau \end{aligned} \quad (5.3)$$

with the definition:

$$G_\mu^o(p) = p^o G_\mu(p) = p^o p_\mu / (p^2 + i\varepsilon). \quad (5.4)$$

The additional factor p^o arises from the symmetrization procedure. $M_{\lambda_{\rho\nu}}(r)$ is the integral kernel which is obtained by performing the integration of the inner loop. It is given by

$$M_{\lambda_{\rho\nu}}(r) = \pi^2 \{ r_\lambda r_\rho r_\nu A^{(1)}(r) + [g_{\lambda\rho} r_\nu + g_{\lambda\nu} r_\rho + g_{\rho\nu} r_\lambda] A^{(2)}(r) + g_{\rho\nu} r_\lambda A^{(3)}(r) \}. \quad (5.5)$$

The functions $A^{(i)}(r)$ are defined in Appendix II.

Equation (5.3) consists of two terms: The first one is a direct term with the integral kernel depending essentially on the difference of the arguments. The second one is an exchange term with the kernel depending on the sum of the arguments.

Now we have to perform the angular momentum reduction of (5.3). It is easy in principle with the formalism developed in Sect. 3, but rather cumbersome in practice. The relevant formulas were collected in I with the exception of terms containing the ε -tensor which arises from the Pauli algebra. They do not contribute for $j=0$ as is shown in Appendix I. After this reduction, the equations take the form:

$$P^2 \varphi_{j's}^{sl}(p_0, p_1) = \sum_{s'l'} \int dq_0 dq_1 q_1^2 K^{sl, s'l'}(p_0, p_1; q_0, q_1) \varphi_{j's}^{s'l'}(q_0, q_1). \quad (5.6)$$

For $j=0$, there are two equations, and four for all other values of j . $s=0$ and $s=1$ couple in such a fashion that the symmetry property (3.6) is preserved.

Each of the integral kernels consists of two terms corresponding to the two terms in (5.3). But we can now reduce the exchange term to a direct one with the help of the symmetry relations (3.6) if we make the substitution $q_0' = -q_0$. An additional factor $(-1)^l$ arises from performing the angular integration.

The complicated system (5.6) cannot be solved analytically. So we have to use some approximation method for the calculation of the eigenvalues. The most simple one is the Fredholm approximation which

requires the calculation of the trace. We will use it here in order to get a first insight. So we get the eigenvalues from:

$$P^2 = \sum_{s,l} \int dp_0 dp_1 p_1^2 K^{sl,sl}(p_0, p_1; p_0, p_1). \quad (5.7)$$

In spite of its appearance, (5.7) is not a linear eigenvalue problem for the mass P^2 of the bound state since K depends also on P^2 in a complicated fashion. It is rather an eigenvalue equation for the coupling constant l^4 which will depend on P^2 as a parameter. The mass will finally be fixed by giving l^4 its physical value. We will now give the equations explicitly for $j=0$. With $P^2 = m^2$ they read:

$$\begin{aligned} m^2 \varphi_{00}^{00}(p_0, p_1) = & [3 - \tfrac{1}{2}T(T+1)] \int dq_0 dq_1 q_1^2 [p_0^2 k_0^{(1)} + p_0 q_0 k_0^{(2)} + p_0 q_0 k_0^{(3)} \\ & + q_0^2 k_0^{(4)} + m p_0 k_0^{(5)} + m p_0 k_0^{(6)} + m q_0 k_0^{(7)} + m q_0 k_0^{(8)} + m^2 k_0^{(9)} + k_0^{(10)}] \varphi_{00}^{00}(q_0, q_1) \\ & + [3 - \tfrac{1}{2}T(T+1)] \int dq_0 dq_1 q_1^2 [-p_0 p_1 k_1^{(1)} - p_0 q_1 k_0^{(2)} \\ & - p_1 q_0 k_0^{(3)} - q_0 q_1 k_1^{(4)} - m p_1 k_1^{(5)} - m q_1 k_1^{(7)}] \varphi_{00}^{11}(q_0, q_1) \\ & + [3 - T(T+1)] \int dq_0 dq_1 q_1^2 [p_0^2 + p_1^2 - \tfrac{1}{4}m^2] k_0^{(e)} \varphi_{00}^{00}(q_0, q_1) \\ & + [3 - T(T+1)] \int dq_0 dq_1 q_1^2 [2 p_1^2] k_1^{(e)} \varphi_{00}^{11}(q_0, q_1) \end{aligned} \quad (5.8)$$

$$\begin{aligned} m^2 \varphi_{00}^{11}(p_0, p_1) = & [3 - \tfrac{1}{2}T(T+1)] \int dq_0 dq_1 q_1^2 [p_0 p_1 k_0^{(1)} + p_1 q_0 k_0^{(2)} + p_0 q_1 k_1^{(3)} \\ & - q_0 q_1 k_1^{(4)} + m p_1 k_0^{(6)} + m q_1 k_1^{(8)}] \varphi_{00}^{00}(q_0, q_1) \\ & + [3 - \tfrac{1}{2}T(T+1)] \int dq_0 dq_1 q_1^2 [-p_1^2 k_1^{(1)} - p_1 q_1 k_0^{(2)} \\ & - \tfrac{1}{2} p_1 q_1 k_0^{(3)} - \tfrac{3}{2} p_1 q_1 k_2^{(3)} - q_1^2 k_1^{(4)} + k_1^{(10)}] \varphi_{00}^{11}(q_0, q_1) \\ & + [3 - T(T+1)] \int dq_0 dq_1 q_1^2 [2 p_0 p_1] k_0^{(e)} \varphi_{00}^{00}(q_0, q_1) \\ & + [3 - T(T+1)] \int dq_0 dq_1 q_1^2 [p_0^2 + p_1^2 - \tfrac{1}{4}m^2] k_1^{(e)} \varphi_{00}^{11}(q_0, q_1). \end{aligned} \quad (5.9)$$

The integral kernels $k^{(i)}$, $i=1, \dots, 10$ are defined in Appendix II as well as the exchange kernel $k^{(e)}$ which depends on $p_0 + q_0$ and $p_1 + q_1$. This can be changed into a dependence on $p_0 - q_0$, $p_1 - q_1$ as discussed above. For $\eta_{CP} = -1$ their contribution is the same as above with $k^{(e)}$ being a direct term now, whereas for $\eta_{CP} = +1$ we have to change the sign. The index l on $k_l^{(i)}$ means the l -th order partial wave projection of $k^{(i)}$

$$k_l^{(i)}(p_0, p_1; q_0, q_1) = \int_{-1}^1 dz P_l(z) k^{(i)}(p; q) \quad (5.10)$$

with $p_1 q_1 z = \mathbf{p} \cdot \mathbf{q}$. The $k_l^{(i)}$ can be calculated straightforwardly from the formulas of Appendix II.

The calculation of the traces of (5.8) and (5.9) can now be carried out along the lines explained in Sect. 4 and in I. In principle, all of the integrals can be performed analytically. Besides elementary functions, dilogarithms and trilogarithms will occur (for their definition see ¹⁹). The result can be written in the form:

$$\begin{aligned} 1 + \frac{1}{16} \left(\frac{\kappa l}{2\pi} \right)^4 \{ [3 - \tfrac{1}{2}T(T+1)] q_0^d(\lambda) \\ + [3 - T(T+1)] q_0^e(\lambda) \} = 0 \end{aligned} \quad (5.11)$$

for (5.8), and similarly for (5.9):

$$\begin{aligned} 1 + \frac{1}{16} \left(\frac{\kappa l}{2\pi} \right)^4 \{ [3 - \tfrac{1}{2}T(T+1)] q_1^d(\lambda) \\ + [3 - T(T+1)] q_1^e(\lambda) \} = 0. \end{aligned} \quad (5.12)$$

The functions $q(\lambda)$ are given explicitly in Appendix III. For $\eta_{CP} = +1$ we have to change the sign of $q_0^e(\lambda)$ and $q_1^e(\lambda)$. In order to get the eigenvalues, we have to add (5.11), (5.12) and (5.1) (only for $\eta_{CP} = -1$). So the eigenvalue equation takes the form:

$$1 + \frac{1}{16} \left(\frac{\kappa l}{2\pi} \right)^4 q_{T, \eta_{CP}}(\lambda) = 0. \quad (5.13)$$

The functions $q_{T, \eta_{CP}}(\lambda)$ are shown in Figure 1. Note especially the large scale of the ordinate. In order to get a mass value in reasonable agreement with experiment, the coupling constant has to be rather small. The value (4.12) is too large and yields a mass value even beyond the physical threshold at $\lambda = 4$. This unphysical behaviour (smaller bound state mass for smaller coupling constant) is due to the fact that we are beyond the threshold for ghost production and have to avoid it by the prin-

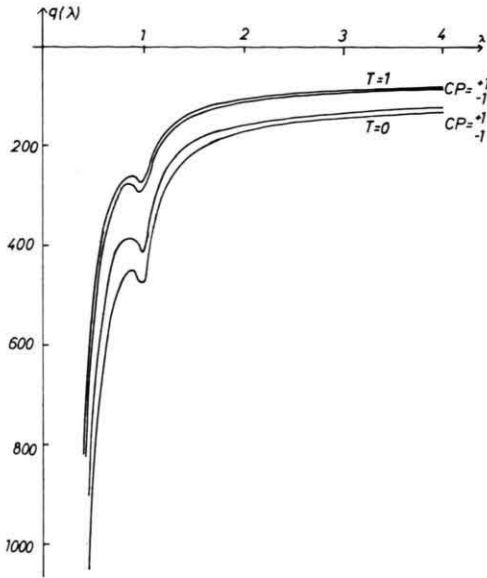


Fig. 1. The functions $q_{T, \eta_{CP}}(\lambda)$ defined by (5.13) for the different values of the isospin T and of the CP-eigenvalues η_{CP} .

cipal value prescription in the last integration as in Section 4.

The splitting between the curves for different η_{CP} with the same isospin is mainly due to the lowest order term (5.1) since the exchange contributions from (5.11) and (5.12) cancel nearly. So the local term (5.1) is only a small correction to the nonlocal ones. The two terms (5.11) and (5.12) are of the same order of magnitude. So it is not a good approximation to neglect the odd amplitudes as it was done in I.

The mass of the particles is fixed by the coupling constant. It would be inconsistent to take the value (4.12) since it was calculated by a different symmetrization procedure. With the average (2.8), the coupling constant was found in ¹⁰ to be

$$(\kappa l/2\pi)^4 = -1.559. \quad (5.14)$$

This would mean an imaginary value of l^2 . So one has to perform a second order calculation for the coupling constant before we can get definite meson masses. Since in first order l^4 is even negative, l^2 will presumably be much smaller than (4.12), so giving a mass of correct order of magnitude. Concerning the norm of the state, we cannot say any-

thing. Neither by the Bethe-Salpeter normalization nor by the functional methods such a calculation has been performed until now. We have not even calculated an appropriate wave function.

6. Conclusions

The result of this paper is not very encouraging: With a coupling constant of conventional magnitude, we find no scalar mesons with a physical reasonable mass value. Before we can draw definite conclusions from this, one should complete the vector meson calculation of I by taking into account also the odd amplitudes. Here we will speculate only a little bit.

There can be several reasons for our failure, taking the validity of the original Eq. (2.5) for granted. Firstly, the Neumann expansion in (2.13) may be not allowed for coupling constants of order 1. A remedy for this is completely unknown at present. Secondly, it may be important to take into account the selfenergy graphs in (2.16). This is confirmed in similar investigations on (2.15) when G is dressed to F by the selfenergy graphs ²⁰. (See also ¹¹.) Thirdly and most probably, the Fredholm approximation may yield wrong results. For the scalar Bethe-Salpeter equation, its validity has been investigated in ⁸. The results were found to be wrong by a factor 2 or 3, being worse for smaller values of the exchanged mass. In (5.3), we have a continuous distribution of the exchanged mass ranging from 0 to ∞ with the spectral function being largest near 0. Additionally, we are now beyond threshold. Of course we could continue the investigations of ⁸ to this energy range also, but unfortunately the exact solutions are known above threshold only for the Wick-Cutkosky model of massless particle exchange ²¹. In this case the Fredholm approximation diverges and yields no meaningful results.

So we can hopefully blame the bad results on the inadequacy of the Fredholm approximation. To get better results, one has to do something better, for example a variational calculation. Investigations in this direction are in progress.

Acknowledgement

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Appendix I

Here we list the formulas needed for performing the angular momentum reduction of (4.1). In practice, one expands $\varphi(p)$ in terms of a \bar{o} -algebra and performs the trace calculation. Then one organizes the equation according to the different Cartesian tensors. The Clebsch-Gordan expansion is then done on each term separately.

We give first the Cartesian tensor and then its contribution to the equation for $\varphi_{jj_3}^{sl}(p_0, p_1)$:

$$\begin{aligned} p_\mu p_\nu: \varphi_{jj_3}^{sl}(p_0, p_1) &= 4\pi \frac{\sqrt{2l+1}}{2j+1} \langle s0l0 | j0 \rangle p_s p_l \int dq_0 dq_1 q_1^2 \varphi_{jj_3}^{j0}(q_0, q_1), \\ P_\mu P_\nu: \varphi_{jj_3}^{sl}(p_0, p_1) &= 4\pi \frac{1}{\sqrt{2j+1}} \delta_{s0} \delta_{lj} m p_j \int dq_0 dq_1 q_1^2 \varphi_{jj_3}^{j0}(q_0, q_1), \\ p_\mu P_\nu: \varphi_{jj_3}^{sl}(p_0, p_1) &= 4\pi (-1)^l \delta_{ls} \delta_{j0} \delta_{j_3 0} m q_s \int dq_0 dq_1 q_1^2 \varphi_{jj_3}^{j0}(q_0, q_1), \\ P_\mu P_\nu: \varphi_{jj_3}^{sl}(p_0, p_1) &= 4\pi \delta_{l0} \delta_{s0} \delta_{j0} \delta_{j_3 0} m^2 \int dq_0 dq_1 q_1^2 \varphi_{jj_3}^{j0}(q_0, q_1), \\ g_{\mu\nu}: \varphi_{jj_3}^{sl}(p_0, p_1) &= 4\pi \delta_{l0} \delta_{sj} \int dq_0 dq_1 q_1^2 \varphi_{jj_3}^{j0}(q_0, q_1), \\ i\varepsilon_{\mu\nu\sigma\rho} P^\sigma: \varphi_{jj_3}^{sl}(p_0, p_1) &= 4\pi \delta_{s1} \delta_{l1} \delta_{j1} \sqrt{\frac{2}{3}} m p_1 \int dq_0 dq_1 q_1^2 \varphi_{jj_3}^{j0}(q_0, q_1). \end{aligned}$$

All these equations are only valid for $j=0, 1$. For all other values of j we get zero.

The corresponding formulas for the nonlocal Eq. (5.3) were already given in I except for the terms containing the ε -tensor. The corresponding Cartesian tensors are:

$$\begin{aligned} -i\varepsilon_{\mu\nu\sigma\lambda} P^\sigma P^\rho q^\lambda q_\nu, & -i\varepsilon_{\mu\nu\sigma\lambda} P^\sigma P^\rho q^\lambda P_\nu, \\ -i\varepsilon_{\mu\nu\sigma\lambda} P^\sigma P^\rho q^\lambda p_\nu, & -i\varepsilon_{\mu\nu\sigma\rho} P^\sigma P^\rho. \end{aligned}$$

They will evidently contribute only for $s=1$ since we are in the rest frame. Furthermore, they will only couple $\varphi_{jj_3}^{1j}$ to $\varphi_{jj_3}^{1j\pm 1}$ and vice versa since the ε -tensor is a pseudotensor and the spherical harmonics belonging to $\varphi_{jj_3}^{1j}$ and $\varphi_{jj_3}^{1j\pm 1}$ have different parity. But for $j=0$, $\varphi_{jj_3}^{1j} \equiv 0$ as is seen from (3.4). So these terms do not contribute for $j=0$. The explicit calculation is facilitated by the use of vector spherical harmonics as it is done in ²². We will not give the results here.

Appendix II

In this appendix we define the integral kernels which occur in (5.8), (5.9). They contain the function $h(p)$ [see (4.5)] and the function

$$h'(p) = g(p_+) f(p_-) - g(p_-) f(p_+)$$

$h(p)$ and $h'(p)$ arise from the external propagators in (5.3). From the integration of the inner loop in (2.16), we have the functions

$$\begin{aligned} A^{(1)}(r) &= \left(-\frac{1}{12} + \frac{1}{2\lambda^2} - \frac{2}{3\lambda^3} + \frac{1}{4\lambda^4} \right) \ln(1-\lambda) + \frac{1}{12} \ln \lambda - \frac{1}{12\lambda} - \frac{13}{24\lambda^2} + \frac{1}{4\lambda^3}, \\ A^{(2)}(r) &= \frac{\kappa^2}{2} \left\{ \left(-\frac{\lambda}{12} + \frac{1}{3} - \frac{1}{2\lambda} + \frac{1}{3\lambda^2} - \frac{1}{12\lambda^3} \right) \ln(1-\lambda) + \left(\frac{\lambda}{12} - \frac{1}{3} \right) \ln \lambda - \frac{1}{12} + \frac{7}{24\lambda} - \frac{1}{12\lambda^2} \right\}, \\ A^{(3)}(r) &= \frac{\kappa^2}{2} \left\{ \left(\frac{\lambda}{6} - \frac{1}{2} + \frac{1}{2\lambda} - \frac{1}{6\lambda^2} \right) \ln(1-\lambda) + \left(-\frac{\lambda}{6} + \frac{1}{2} \right) \ln \lambda + \frac{1}{6} - \frac{1}{6\lambda} \right\} \end{aligned}$$

with $\lambda = r^2/\kappa^2$. Omitting a factor $4l^4/(2\pi)^6$, common to all integral kernels $k^{(i)}(p; q)$, they are given by:

$$\begin{aligned} k^{(1)}(p; q) &= h(p) \{ A^{(1)}(p-q) [4(p \cdot (p-q))^2 + 2(\frac{1}{4}m^2 - p^2)p \cdot (p-q)] \\ &\quad + A^{(2)}(p-q) [8p \cdot (p-q) + m^2] + A^{(3)}(p-q) [2p \cdot (p-q) + \frac{1}{2}m^2] \} \\ &\quad + h'(p) \{ A^{(1)}(p-q) [2p \cdot (p-q) P \cdot (p-q) + (\frac{1}{4}m^2 - p^2) P \cdot (p-q)] \\ &\quad + A^{(2)}(p-q) [4P \cdot p - 2P \cdot q] + A^{(3)}(p-q) [P \cdot p] \}, \end{aligned}$$

$$\begin{aligned}
k^{(2)}(p; q) &= h(p) \{ A^{(1)}(p-q) [-4(p \cdot (p-q))^2 - 2(\frac{1}{4}m^2 - p^2)p \cdot (p-q)] \\
&\quad + A^{(2)}(p-q) [-2p^2 - \frac{1}{2}m^2] + A^{(3)}(p-q) [-p^2 - \frac{1}{4}m^2] \} \\
&\quad + h'(p) \{ A^{(1)}(p-q) [-2p \cdot (p-q)P \cdot (p-q) - (\frac{1}{4}m^2 - p^2)P \cdot (p-q)] \\
&\quad + A^{(2)}(p-q) [-2P \cdot p] + A^{(3)}(p-q) [-P \cdot p] \}, \\
k^{(3)}(p; q) &= h(p) \{ A^{(1)}(p-q) [-2(\frac{1}{4}m^2 - p^2)p \cdot (p-q)] \\
&\quad + A^{(2)}(p-q) [2p^2 - \frac{1}{2}m^2] + A^{(3)}(p-q) [p^2 - \frac{1}{4}m^2] \} \\
&\quad + h'(p) \{ A^{(1)}(p-q) [(p^2 - \frac{1}{4}m^2)P \cdot (p-q)] \}, \\
k^{(4)}(p; q) &= h(p) \{ A^{(1)}(p-q) [2(-p^2 + \frac{1}{4}m^2)p \cdot (p-q)] \} \\
&\quad + h'(p) \{ A^{(1)}(p-q) [(-p^2 + \frac{1}{4}m^2)P \cdot (p-q)] \}, \\
k^{(5)}(p; q) &= h(p) \{ A^{(1)}(p-q) [-p \cdot (p-q)P \cdot (p-q)] \\
&\quad + A^{(2)}(p-q) [-2P \cdot p + P \cdot q] + A^{(3)}(p-q) [-P \cdot p + \frac{1}{2}P \cdot q] \} \\
&\quad + h'(p) \{ A^{(1)}(p-q) [-\frac{1}{2}(P \cdot (p-q))^2] \\
&\quad + A^{(2)}(p-q) [-p^2 - \frac{1}{4}m^2] + A^{(3)}(p-q) [-\frac{1}{2}p^2 - \frac{1}{8}m^2] \}, \\
k^{(6)}(p; q) &= h'(p) \{ A^{(2)}(p-q) [2p(p-q) - p^2 + \frac{1}{4}m^2] + A^{(3)}(p-q) [p \cdot (p-q) - p^2 + \frac{1}{4}m^2] \}, \\
k^{(7)}(p; q) &= h(p) \{ A^{(1)}(p-q) [p \cdot (p-q)P \cdot (p-q)] + A^{(2)}(p-q) [P \cdot p] + A^{(3)}(p-q) [\frac{1}{2}P \cdot p] \} \\
&\quad + h'(p) \{ A^{(1)}(p-q) [\frac{1}{2}(P \cdot (p-q))^2] + A^{(2)}(p-q) [-p^2 + \frac{1}{4}m^2] \\
&\quad + A^{(3)}(p-q) [-\frac{1}{2}p^2 + \frac{1}{8}m^2] \}, \\
k^{(8)}(p; q) &= h'(p) \{ A^{(2)}(p-q) [p^2 - \frac{1}{4}m^2] + A^{(3)}(p-q) [\frac{1}{2}p^2 - \frac{1}{8}m^2] \}, \\
k^{(9)}(p; q) &= h(p) \{ A^{(2)}(p-q) [-p \cdot (p-q)] \} \\
&\quad + h'(p) \{ A^{(2)}(p-q) [-P \cdot (p-q)] + A^{(3)}(p-q) [-\frac{1}{2}P \cdot (p-q)] \}, \\
k^{(10)}(p; q) &= h(p) \{ A^{(2)}(p-q) [(-2p^2 + \frac{1}{2}m^2)p \cdot (p-q)] \} \\
&\quad + h'(p) \{ A^{(2)}(p-q) [(-p^2 + \frac{1}{4}m^2)P \cdot (p-q)] \}.
\end{aligned}$$

For the exchange term, we have

$$\begin{aligned}
k^{(e)}(p; q) &= h(p) \{ A^{(1)}(p+q) [-\frac{1}{2}p \cdot (p+q)(p+q)^2] \\
&\quad + A^{(2)}(p+q) [-3p \cdot (p+q)] + A^{(3)}(p+q) [-\frac{1}{2}p \cdot (p+q)] \} \\
&\quad + h'(p) \{ A^{(1)}(p+q) [-\frac{1}{2}P \cdot (p+q)(p+q)^2] \\
&\quad + A^{(2)}(p+q) [-3P \cdot (p+q)] + A^{(3)}(p+q) [-\frac{1}{2}P \cdot (p+q)] \}.
\end{aligned}$$

By taking the “diagonal” terms $k_l^{(i)}(p_0, p_1; p_0, p_1)$ which are needed for calculating the trace, the kernels simplify considerably: All terms involving $P \cdot (p-q)$ will vanish because we are in the rest frame $P = (m, 0, 0, 0)$. For the exchange term $k^{(e)}$ this is true only after making use of the symmetry of the wavefunction.

Appendix III

In this appendix we list the functions $q(\lambda)$ which occur in (5.11) and (5.12). Besides elementary functions, they involve the dilogarithm $\mathcal{L}_2(\lambda)$, the trilogarithm $\mathcal{L}_3(\lambda)$ (for their definition see¹⁹), and the functions

$$F(\lambda) = \int_0^1 \frac{dx}{(1-x)\lambda-1} \mathcal{L}_2\left(-\frac{x^2}{1-x}\right), \quad G(\lambda) = \lambda \int_0^1 \frac{dx}{x\lambda-1} \ln(1-x+x^2).$$

For (5.11) we have

$$\begin{aligned}
 q_0^d(\lambda) = & \frac{16\pi^2}{81}\lambda + \left(-\frac{7}{18} + \frac{\pi}{9\sqrt{3}} + \frac{35\pi^2}{27}\right) + \left(\frac{403}{18} + \frac{31\pi}{18\sqrt{3}} + \frac{65\pi^2}{36} + \frac{2}{3}F(1)\right)\frac{1}{\lambda} \\
 & + \left(\frac{397}{108} - \frac{37\pi}{12\sqrt{3}} + \frac{29\pi^2}{24}\right)\frac{1}{\lambda^2} + \left(\frac{1}{18} - \frac{31\pi}{18\sqrt{3}} - \frac{\pi^2}{2}\right)\frac{1}{\lambda^3} - \frac{\pi}{9\sqrt{3}}\frac{1}{\lambda^4} \\
 & + \left(\frac{2\pi}{3\sqrt{3}} - \frac{2\pi^2}{27}\right)\frac{1}{\lambda-1} + \frac{2\pi^2}{27}\frac{1}{(\lambda-1)^2} + \left(\frac{1}{9} + \frac{101}{18\lambda} + \frac{29}{3\lambda^2} + \frac{101}{18\lambda^3} + \frac{1}{9\lambda^4}\right)\ln|1-\lambda| \\
 & + \left(-\frac{1}{2} - \frac{231}{2\lambda} + \frac{3}{\lambda^2}\right)\ln|\lambda| + \left(-\frac{\lambda}{4} - \frac{15}{8}\right)\ln^2|\lambda| + \frac{7}{3\lambda}F(\lambda) \\
 & + \left(-\frac{1}{9}\lambda - \frac{1}{6} - \frac{26}{9\lambda} + \frac{80}{9\lambda^2} - \frac{4}{3\lambda^3} - \frac{1}{6\lambda^4} - \frac{1}{9\lambda^5} - \frac{2}{3}\frac{1}{\lambda-1} + \frac{2}{3}\frac{1}{(\lambda-1)^2}\right)G(\lambda) \\
 & + \left(\frac{41}{9\lambda} - \frac{35}{3\lambda^2} + \frac{3}{\lambda^3} + \frac{2}{3}\frac{1}{\lambda-1} - \frac{2}{3}\frac{1}{(\lambda-1)^2}\right)\mathcal{L}_2(\lambda) + \left(-\frac{5}{3}\left(\frac{\lambda-1}{\lambda}\right)^4\right. \\
 & \left.- \frac{2}{9}\frac{(\lambda-1)^6}{\lambda^5}\right)\mathcal{L}_2\left(\frac{\lambda}{\lambda-1}\right) + \left(-\frac{1}{2}\lambda - \frac{15}{4} - \frac{27}{2\lambda} - \frac{29}{4\lambda^2} + \frac{3}{\lambda^3}\right)\mathcal{L}_2(1+\lambda) \\
 & - \frac{21}{2\lambda^2}\ln|\lambda|\mathcal{L}_2(-\lambda) + \frac{21}{\lambda^2}\mathcal{L}_3(-\lambda), \\
 q_0^e(\lambda) = & -\frac{\pi^2}{54}\lambda + \left(\frac{2}{3} + \frac{\pi}{3\sqrt{3}} - \frac{23\pi^2}{36}\right) + \left(-\frac{22}{3} - \frac{\pi}{3\sqrt{3}} - \frac{13\pi^2}{9}\right)\frac{1}{\lambda} + \left(\frac{163}{36} - \frac{7\pi}{4\sqrt{3}} - \frac{11\pi^2}{36}\right)\frac{1}{\lambda^2} \\
 & + \left(\frac{1}{6} + \frac{\pi}{3\sqrt{3}}\right)\frac{1}{\lambda^3} - \frac{\pi}{3\sqrt{3}}\frac{1}{\lambda^4} + \left(\frac{1}{3} + \frac{3}{2\lambda} - \frac{11}{3\lambda^2} + \frac{3}{2\lambda^3} + \frac{1}{3\lambda^4}\right)\ln|1-\lambda| \\
 & + \left(\frac{1}{3} + \frac{17}{6\lambda}\right)\ln|\lambda| + \left(\frac{1}{6}\lambda + \frac{3}{4}\right)\ln^2|\lambda| + \frac{1}{\lambda}F(\lambda) \\
 & + \left(-\frac{11}{6\lambda^2} - 3\frac{(\lambda-1)^2}{\lambda^3} - \frac{3}{2}\left(\frac{\lambda-1}{\lambda}\right)^4 - \frac{1}{3}\frac{(\lambda-1)^6}{\lambda^5}\right)G(\lambda) \\
 & + \left(\frac{11}{6\lambda^2} + 3\frac{(\lambda-1)^2}{\lambda^3}\right)\mathcal{L}_2(\lambda) + \left(-3\left(\frac{\lambda-1}{\lambda}\right)^4 - \frac{2}{3}\frac{(\lambda-1)^6}{\lambda^5}\right)\mathcal{L}_2\left(\frac{\lambda}{\lambda-1}\right) \\
 & + \left(\frac{1}{3}\lambda + \frac{3}{2} + \frac{3}{\lambda} + \frac{11}{6\lambda^2}\right)\mathcal{L}_2(1+\lambda) + \frac{1}{\lambda^2}\ln|\lambda|\mathcal{L}_2(-\lambda) + \frac{2}{\lambda^2}\mathcal{L}_3(-\lambda).
 \end{aligned}$$

Similarly, we have for (5.12):

$$\begin{aligned}
 q_1^d(\lambda) = & -\frac{55\pi^2}{324}\lambda^2 + \left(-\frac{5\pi}{18\sqrt{3}} + \frac{121\pi^2}{324}\right)\lambda + \left(-\frac{5}{12} + \frac{17\pi}{36\sqrt{3}} + \frac{529\pi^2}{162}\right) \\
 & + \left(-\frac{8045}{216} - \frac{1477\pi}{36\sqrt{3}} - \frac{626\pi^2}{81} + \frac{10}{3}F(1)\right)\frac{1}{\lambda} + \left(-\frac{170}{27} - \frac{91\pi}{18\sqrt{3}} - \frac{457\pi^2}{162}\right)\frac{1}{\lambda^2} \\
 & + \left(\frac{1}{9} - \frac{\pi}{3\sqrt{3}} - \frac{71\pi^2}{108}\right)\frac{1}{\lambda^3} - \frac{4\pi}{9\sqrt{3}}\frac{1}{\lambda^4} + \left(-\frac{55}{36} + \frac{253\pi}{36\sqrt{3}} - \frac{13\pi^2}{27}\right)\frac{1}{\lambda-1} \\
 & + \left(-\frac{55\pi}{18\sqrt{3}} + \frac{13\pi^2}{54} + 2F(1)\right)\frac{1}{(\lambda-1)^2} + \left(-\frac{31\pi^2}{162} - \frac{2}{3}F(1)\right)\frac{1}{(\lambda-1)^3} \\
 & + \left(-\frac{5}{18}\lambda - \frac{101}{36} - \frac{301}{36\lambda} + \frac{143}{9\lambda^2} - \frac{97}{9\lambda^3} + \frac{4}{9\lambda^4} + \frac{43}{18}\frac{1}{\lambda-1}\right)\ln|1-\lambda| \\
 & + \left(\frac{5}{18}\lambda + \frac{9}{4} + \frac{145}{4\lambda} + \frac{3}{2\lambda^2}\right)\ln|\lambda| + \left(\frac{5}{36}\lambda^2 - \frac{11}{36}\lambda - \frac{7}{2}\right)\ln^2|\lambda| \\
 & + \left(-\frac{16}{3\lambda} + \frac{64}{3\lambda^2}\right)\ln|\lambda|\ln|1+\lambda| + \left(-\frac{4}{3} - \frac{14}{3\lambda} - \frac{2}{\lambda-1} - \frac{4}{3}\frac{1}{(\lambda-1)^2} + \frac{2}{3}\frac{1}{(\lambda-1)^3}\right)F(\lambda)
 \end{aligned}$$

$$\begin{aligned}
& + \left(4 + \frac{16}{9\lambda} + \frac{5}{9\lambda^2} - \frac{4}{\lambda^3} - \frac{2}{\lambda-1} + \frac{3}{2} \frac{1}{(\lambda-1)^2} - \frac{31}{18} \frac{1}{(\lambda-1)^3} \right) (G(\lambda) - \mathcal{L}_2(\lambda)) \\
& + (\lambda-1)^3 \left(\frac{5}{18\lambda} + \frac{2}{9\lambda^2} - \frac{49}{18\lambda^3} + \frac{14}{9\lambda^4} + \frac{4}{9\lambda^5} \right) \left(G(\lambda) + 2\mathcal{L}_2 \left(\frac{\lambda}{\lambda-1} \right) \right) \cdot (G(\lambda) - \mathcal{L}_2(\lambda)) \\
& + \left(\frac{5}{18} \lambda^2 - \frac{11}{18} \lambda - 3 + \frac{238}{9\lambda} + \frac{809}{18\lambda^2} - \frac{209}{18\lambda^3} \right) \mathcal{L}_2(1+\lambda) \\
& + \left(\frac{4}{3\lambda} + \frac{12}{\lambda^2} + \frac{8}{3\lambda^3} \right) \ln|\lambda| \mathcal{L}_2(-\lambda) + \left(-\frac{16}{3\lambda} - \frac{40}{3\lambda^2} - \frac{4}{\lambda^3} \right) \mathcal{L}_3(-\lambda) \\
q_1^e(\lambda) = & -\frac{11\pi^2}{108} \lambda + \left(-\frac{\pi}{6\sqrt{3}} - \frac{\pi^2}{36} \right) + \left(\frac{3}{4} - \frac{\pi}{12\sqrt{3}} - \frac{\pi^2}{12} \right) \frac{1}{\lambda} + \left(\frac{10}{9} + \frac{\pi}{4\sqrt{3}} - \frac{\pi^2}{36} \right) \frac{1}{\lambda^2} \\
& + \left(-\frac{1}{12} - \frac{5\pi}{12\sqrt{3}} - \frac{\pi^2}{12} \right) \frac{1}{\lambda^3} + \frac{\pi}{6\sqrt{3}} \frac{1}{\lambda^4} + \left(\frac{\pi}{2\sqrt{3}} - \frac{\pi^2}{18} \right) \frac{1}{\lambda-1} \\
& + \frac{\pi^2}{18} \frac{1}{(\lambda-1)^2} + \left(-\frac{1}{6} + \frac{7}{4\lambda} - \frac{11}{3\lambda^2} + \frac{7}{4\lambda^3} - \frac{1}{6\lambda^4} \right) \ln|1-\lambda| \\
& + \left(\frac{1}{6} - \frac{13}{12\lambda} + \frac{1}{2\lambda^2} \right) \ln|\lambda| + \left(\frac{\lambda}{12} + \frac{1}{4} \right) \ln^2|\lambda| + \frac{1}{\lambda} F(\lambda) \\
& + \frac{1}{6} \frac{\lambda^2 + \lambda + 1}{\lambda^2} \left(\frac{(\lambda-1)^4}{\lambda^3} + \frac{1}{(\lambda-1)^2} \right) G(\lambda) - \frac{1}{6} \frac{\lambda^2 + \lambda + 1}{\lambda^2} \mathcal{L}_2(\lambda) \\
& + \left(\frac{1}{6} \lambda + \frac{1}{2} + \frac{1}{6\lambda^2} + \frac{1}{2\lambda^3} \right) \mathcal{L}_2(1+\lambda) + \frac{(\lambda-1)^3 (\lambda^3-1)}{3\lambda^5} \mathcal{L}_2 \left(\frac{\lambda}{\lambda-1} \right) \\
& - \frac{1}{\lambda^2} \ln|\lambda| \mathcal{L}_2(-\lambda) - \frac{2}{\lambda^2} \mathcal{L}_3(-\lambda).
\end{aligned}$$

In order to be sure about the correctness of the tedious calculations, one has a number of tests. The first is the cancellation of the divergent terms one meets during the calculation. The next is the absence of a singularity in $q(\lambda)$ at $\lambda=1$. To check this, one uses the expansions

$$\begin{aligned}
F(\lambda) &= F(1) + (\lambda-1) \left[\frac{2\pi}{\sqrt{3}} - \frac{\pi^2}{9} - F(1) \right] + (\lambda-1)^2 \left[1 - \frac{3\pi}{\sqrt{3}} + \frac{\pi^2}{6} + F(1) \right] + \dots \\
&\quad + \ln|1-\lambda| [-(\lambda-1)^2 + 2(\lambda-1)^3 + \dots] \\
G(\lambda) &= \mathcal{L}_2(\lambda) + \frac{\pi^2}{9} + (\lambda-1) \frac{\pi}{\sqrt{3}} + (\lambda-1)^2 \left[\frac{1}{2} - \frac{\pi}{2\sqrt{3}} \right] + \dots \\
&\quad + \ln|1-\lambda| [-(\lambda-1)^2 + (\lambda-1)^3 + \dots].
\end{aligned}$$

Finally, $q(\lambda)$ has to vanish for $\lambda \rightarrow \infty$. The asymptotic behaviour of the nonelementary functions is:

$$\begin{aligned}
\mathcal{L}_2(\lambda) &= -\frac{1}{2} \ln^2 \lambda + \frac{\pi^2}{3} - \frac{1}{\lambda} + \dots, \\
\mathcal{L}_2 \left(\frac{\lambda}{\lambda-1} \right) &= \frac{\pi^2}{6} + \frac{1}{\lambda} - \frac{1}{4\lambda^2} + \dots + \ln \lambda \cdot \left(\frac{1}{\lambda} + \frac{1}{2\lambda^2} + \dots \right), \\
\mathcal{L}_3(-\lambda) &= -\frac{1}{6} \ln^3 \lambda - \frac{\pi^2}{6} \ln \lambda - \frac{1}{\lambda} - \dots, \\
G(\lambda) &= -\frac{\pi^2}{18} + \left(-1 + \frac{\pi}{\sqrt{3}} \right) \frac{1}{\lambda} + \left(\frac{3}{4} + \frac{\pi}{2\sqrt{3}} \right) \frac{1}{\lambda^2} + \dots + \ln \lambda \cdot \left(-\frac{1}{\lambda} + \frac{1}{2\lambda^2} + \dots \right), \\
F(\lambda) &= -\frac{1}{6\lambda} \ln^3 \lambda + \dots
\end{aligned}$$

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